ORKUSTOFNUN
NATIONAL ENERGY AUTHORITY

RAUNVÍSINDASTOFNUN
HÁSKÓLANS
Science institute - University of Iceland

GROUNDWATER FLOW

Treatment of timederivative and calculation of flow in the Galerkin finite element methods

Snorri Páll Kjaran

National Energy Authority
Electric Power Division

Sven T. Sigurdsson

Science Institute -
University of Iceland

OS79043/ROD15

RH-79-12

Reykjavík, October 1979

GROUNDWATER FLOW

Treatment of timederivative and calculation of flow in the Galerkin finite element methods

Snorri Páll Kjaran

National Energy Authority
Electric Power Division

Sven T. Sigurdsson

Science Institute -
University of Iceland

OS79043/ROD15 RH-79-12

Reykjavík, October 1979

ABSTRACT

Two techniques connected with the use of the finite element Galerkin method for solving the linear parabolic differential equation describing unsteady groundwater flow in an anisotropic nonhomogenous aquifer are introduced. The first is a mode superposition technique for dealing with the timederivative which involves computing the smallest eigenvalues and associated eigenvectors of the matrices arising from the Galerkin method. It is shown how such a technique allows us to interpret the response of the groundwaterlevel to input in terms of parallel linear reservoirs. It is further argued that if properly implemented, the technique will have computational advantages over standard finite difference methods f.ex. in the case when the input function is constant over relatively large time subintervals. The second is a technique based on socalled generalized flow formulae for calculating flow values across external or internal boundaries, posterior to obtaining the groundwaterlevel values. The implementation of the technique in the case of linear triangular elements on an irregular grid is discussed. It is finally argued from simplified cases that, apart from guaranteeing match with prescribed input, the technique may often be expected to give more accurate flow values than those obtained directly from the groundwater gradients.

CONTENTS

LIST OF FIGURES

1 INTRODUCTION

The linear parabolic differential equation describing unsteady groundwater flow in an anisotropic nonhomogenecus aquifer has been solved by various authors using the finite element Galerkin method for the space variables. This paper deals with two techniques connected with the implementation of this method. It is an extended and revised version of a paper presented by the authors at the Second International Conference on Finite Elements in Water Resources in London in 1978 (see Kjaran and Sigurdsson (1978)).

We discuss, firstly, a mode superposition technique for dealing with the timederivative. The technique involves computing the smallest eigenvalues and associated eigenvectors of the matrices arising from the Galerkin method. This can be done effectively with the subspace iteration algorithm which has been developed for structural problems. An advantage of this technique from a hydrological standpoint is that it allows us to interpret the response of the approximate groundwaterlevel to infiltration or pumping in terms of simple parallel linear reservoirs. The eigenvalues can be associated with the timeconstants of the reservoirs and the inflow to each reservoir is determined by the eigenvectors. This interpretation carries over from the exact theoretical groundwaterlevel but whereas the response of this level can be shown to be equivalent to the response of inifinitely many parallel linear reservoirs with definite timeconstants, the technique presented in this paper only includes approximations to the constants of the most delayed reservoirs and treats the remaining inflow without time-delay. From a computational standpoint the technique appears to offer an attractive alternative to the use of finite difference methods in cases where the input function is piecewise constant (or can be approximated by low-order polynomials) over relatively large subintervals of the total timeinterval of interest or where groundwaterlevel values are only required at relatively few timepoints compared with the total time interval of interest.

It is sometimes of equal importance to be able to obtain good approximations to flow values across external or internal boundaries as to actual groundwater levels. It is well known that, if these flow values are obtained directly from the ground-

water gradients, matching with prescribed infiltration or inflow will in general not be attained. It is, however, possible to make use of difference approximations to normal gradients, that are inherent to the Galerkin method and sometimes referred to as generalized flow formulae, to overcome this matching problem. We consider, secondly, in this paper a realization of this type of approximation in the case of linear triangular elements in an irregular region. Although it is difficult to prove in general that the resulting flow values will be more accurate than gradient flow values, it can be shown that they correspond to improved order difference approximations for the flow in a special regular case arising when we have uniform triangularization of a rectangular region. Furthermore, there is experimental evidence of the usefulness of analogous gradient formulae in two-dimensional field problems with regular rectangular elements.

2 THE BASIC EQUATION AND ITS APPROXIMATION BY THE GALERKIN FINITE ELEMENT METHOD

In this section we present the basic equation for groundwater flow which we are concerned with in this paper and its approximation by the Galerkin finite element method using linear triangular elements. The Galerkin method has been used by many authors, in one form or another, for solving groundwater problems, see f.ex. Neuman and Witherspoon (1970), Pinder and Frind (1972), and Pinder and Gray (1974). In this section we, therefore, restrict ourselves to a brief development of the method in the form that is relevant for the special techniques to be presented in the following sections. A more general treatment of the method can be found in a variety of texts f.ex. Zienkiewicz (1971), Strang and Fix (1973), Mitchell and Wait (1977), or Pinder and Gray (1977).

The differential equation governing the groundwaterlevel, h, in an anisotropic, nonhomogeneous aquifer is given by (c.f. figure 1):

$$S\frac{\partial h}{\partial t} = Lh + R(x,y,t) \text{ in } \Omega \qquad (1)$$

$$h = g \text{ on } \partial\Omega_1 \qquad (2)$$

$$-Nh = f \text{ on } \partial\Omega_2$$

$$h(x,y,t_o) = \hat{h}(x,y) \quad \text{in} \quad \Omega \tag{3}$$

where the operators L and N are defined as:

$$L = \frac{\partial}{\partial x}\left(T_{xx}\frac{\partial}{\partial x} + T_{xy}\frac{\partial}{\partial y}\right) + \frac{\partial}{\partial y}\left(T_{yy}\frac{\partial}{\partial y} + T_{yx}\frac{\partial}{\partial x}\right) \tag{4}$$

$$N = \underset{\sim}{n}\cdot\left(T_{xx}\frac{\partial}{\partial x} + T_{xy}\frac{\partial}{\partial y}\ ,\ T_{yy}\frac{\partial}{\partial y} + T_{yx}\frac{\partial}{\partial x}\right). \tag{5}$$

In the isotropic case (4) and (5) reduce to

$$L = \nabla\cdot(T\nabla) \tag{6}$$

$$N = T\frac{\partial}{\partial n}. \tag{7}$$

R is some source function (infiltration, pumping, recharge) and g and f are given timeindependent functions. S denotes the storage coefficient and T the transmissivity.

In order to obtain the Galerkin finite element approximation to the solution of equation (1) to (5) the region Ω is subdivided into triangular elements (cf. figure 2) and the nodal points are numbered, starting with points in Ω and on $\partial\Omega_2$ (assume the total number of these to be N), and finishing with points on $\partial\Omega_1$ (assume the total number of these to be M-N). We associate with nodal point i the piecewise linear pyramid function $\Psi_i(x,y)$, which takes the value 1 at the point i and the value 0 outside the adjacent triangles (the shaded region in figure 2). We then seek to obtain a continuous and piecewise linear approximate solution in the form:

$$\bar{h}(x,y,t) = \sum_{i=1}^{N} h_i(t)\,\psi_i(x,y) + \sum_{i=N+1}^{M} g_i\psi_i(x,y). \tag{8}$$

The coefficients $h_i(t)$ or g_i are the values of $\bar{h}$ at point i, the constant coefficients g_i are obtained from the prescribed boundary condition on $\partial\Omega_1$, whereas the time dependent coefficients $h_i(t)$ are to be determined. Thus we have ensured that $\bar{h}$ approximately satisfies the condition (2) on $\partial\Omega_1$. The flow condition (3) on $\partial\Omega_2$, on the other hand, is only indirectly imposed on $\bar{h}$ as shown below.

Considering first the case of isotropic flow, the Galerkin conditions for determining $h_i(t)$ can formally be described as:

$$\iint_{\Omega}\{S\frac{\partial \bar{h}}{\partial t} - \nabla(T\nabla\bar{h}) - R\}\psi_i dxdy = 0,\ i=1,2,\ldots,N. \qquad (9)$$

However, since $\bar{h}$ is only piecewise once differentiable in x and y, these conditions have no proper meaning until they have been transformed into the corresponding weak form by integrating by parts using Green's theorem. Equation (9) can then be rewritten as:

$$C\frac{d\underset{\sim}{h}}{dt} = -B\underset{\sim}{h} + \underset{\sim}{b} \qquad (10)$$

where $\underset{\sim}{h}(t) = [h_i(t)]_N$

$$C = [c_{ij}]_{NxN} = [\iint_{\Omega} S\psi_j\psi_i dxdy]_{NxN} \qquad (11)$$

$$B = [b_{ij}]_{NxN} = [\iint_{\Omega}(T\nabla\psi_j\cdot\nabla\psi_i)dxdy]_{NxN} \qquad (12)$$

$$\underset{\sim}{b}(t) = [b_i(t)]_N = [\iint_{\Omega} R\psi_i dxdy - \iint_{\Omega}(T\sum_{j=N+1}^{M} g_j\nabla\psi_j\cdot\nabla\psi_i)dxdy$$

$$-\oint_{\partial\Omega_2} f\psi_i ds]_N \qquad (13)$$

The presence of the line-integral in (13) and the absence of a corresponding integral in (12) indirectly impose on $\bar{h}$ the flow condition (3) on $\partial\Omega_2$. Equation (10) represents a system of N linear ordinary differential equations that can be solved for given initial conditions $\underset{\sim}{h}(t_o) = \underset{\sim}{\hat{h}}$ (cf. condition(3)). For the steady state problem with $\frac{d\underset{\sim}{h}}{dt} = 0$ and $\underset{\sim}{b}(t) = \underset{\sim}{\bar{b}}$, corresponding to the replacement of the sourcefunction R(x,y,t) by a long-term average source level $\bar{R}(x,y)$, it reduces to a system of N linear equations. In the evaluation of the integrals in (11) to (13) we use the approximations that S and T are constant within each triangular element and that R is continous in Ω and linear within each triangular element, taking prescribed values at the nodal points, referred to as R_i. We can also include in the infiltration term, R, Dirac delta functions, $-Q\delta(x,y;\xi,\eta)$, corresponding to a pumping at the point (ξ,η). Finally, we use the approximation that f is linear between nodal points on $\partial\Omega_2$, taking on prescribed flow values at the nodal points, referred to as f_{i-} or f_{i+} depending on whether we are

considering the flow value just before or just after the nodal point as we pass along $\partial\Omega_2$ anticlockwise (cf. figure 2).

The differential equation system (10) is most readily obtained by assembling it from contributions of each triangular element. These contributions can, in turn, be obtained by applying the Galerkin method to such an element and imposing arbitrary flow conditions round the whole boundary (cf. figure 3). The resulting system corresponding to (10) is, if we assume that we have a pumping Q at point P as well as a continuous infiltration:

$$\frac{SA}{12}\begin{Bmatrix}2&1&1\\1&2&1\\1&1&2\end{Bmatrix}\frac{d}{dt}\begin{Bmatrix}h_i\\h_j\\h_k\end{Bmatrix} =$$

$$-\frac{T}{SA}\begin{Bmatrix}2l_{jk}^2 & l_{ij}^2-l_{jk}^2-l_{ki}^2 & l_{ki}^2-l_{ij}^2-l_{jk}^2\\ l_{ij}^2-l_{jk}^2-l_{ki}^2 & 2l_{ki}^2 & l_{jk}^2-l_{ki}^2-l_{ij}^2\\ l_{ki}^2-l_{ij}^2-l_{ik}^2 & l_{jk}^2-l_{ki}^2-l_{ij}^2 & 2\,l_{ij}^2\end{Bmatrix}\begin{Bmatrix}h_i\\h_j\\h_k\end{Bmatrix}$$

$$+\frac{A}{12}\begin{Bmatrix}2&1&1\\1&2&1\\1&1&2\end{Bmatrix}\begin{Bmatrix}R_i\\R_j\\R_k\end{Bmatrix} -Q\begin{Bmatrix}\alpha_i\\\alpha_j\\\alpha_k\end{Bmatrix} \qquad (14)$$

$$-\frac{1}{6}\begin{Bmatrix}l_{ki}(f_{k+}+2f_{i-})+l_{ij}(2f_{i+}+f_{j-})\\ l_{ij}(f_{i+}+2f_{j-})+l_{jk}(2f_{j+}+f_{k-})\\ l_{jk}(f_{j+}+2f_{k-})+l_{ki}(2f_{k+}+f_{i-})\end{Bmatrix}$$

Here A denotes the area of the element and l_{ij} the length of the edge between points i and j.

$$\alpha_i = \frac{l_{pq}}{l_{iq}}\text{ , similarly for }\alpha_j\text{ and }\alpha_k.$$

When assembling the element contributions we let flow terms across common boundaries of adjacent elements cancel out. In the final assembled system we are thus only left with flow terms on the outer boundaries $\partial\Omega_1$ and $\partial\Omega_2$. On $\partial\Omega_2$ the flow

terms are prescribed, whereas on $\partial\Omega_1$ h_i are known and we can drop the corresponding equations from the system. The resulting system will then be identical to that in (10). It should further be noted that if we sum the individual equations in (14) we get that:

$$SA\frac{d}{dt}\left(\frac{h_i+h_j+h_k}{3}\right) = A\,\frac{R_i+R_j+R_k}{3} - Q$$

$$-l_{ij}\left(\frac{f_{i+}+f_{j-}}{2}\right) - l_{jk}\left(\frac{f_{j+}+f_{k-}}{2}\right) - l_{ki}\left(\frac{f_{k+}+f_{i-}}{2}\right).$$

Thus mass is conserved within the element and the same will hold true for any assembled subregion and in the end for the total region Ω.

Considering, finally, the case of anisotropic flow, the change in equations (10) to (13) is that in equation (12) we now have that:

$$b_{ij} = \iint_{\tilde{\Omega}} \left[T_{xx}\,\frac{\partial\psi_j}{\partial x}\,\frac{\partial\psi_i}{\partial x} + T_{yy}\,\frac{\partial\psi_j}{\partial y}\,\frac{\partial\psi_i}{\partial y} + T_{xy}\,\frac{\partial\psi_j}{\partial y}\,\frac{\partial\psi_i}{\partial x} + T_{yx}\,\frac{\partial\psi_j}{\partial x}\,\frac{\partial\psi_i}{\partial y}\right]dxdy$$

(cf. equation (4)). However, if we approximate T_{xx}, T_{yy}, T_{xy}, and T_{yx} with constants within each triangular element, this can be brought within the framework of equation (14) if we determine the eigenvalues, λ_1 and λ_2, and corresponding unit eigenvectors (principal directions of the anisotropy) (c_{11}, c_{12}) and (c_{21}, c_{22}) for the matrix

$$\begin{bmatrix} T_{xx} & T_{xy} \\ T_{yx} & T_{yy} \end{bmatrix}$$

Assuming that $T_{xy} = T_{yx}$ and $T_{xx}\,T_{yy} > T_{xy}^2$, so that the eigenvalues are real and positive and the eigenvectors orthonormal, we introduce the following change of coordinates:

$$\xi = \frac{1}{\sqrt{\lambda_1}}\left[c_{11}\,x + c_{12}y\right], \quad \eta = \frac{1}{\sqrt{\lambda_2}}\left[c_{21}x + c_{22}y\right]. \quad (15)$$

If the lengths l_{ij}, l_{jk}, and l_{ki} in the first matrix on the

RHS of equation (14) are recalculated in this new coordinate system and T is replaced by $\lambda_1\lambda_2$ the triangular element contribution to the final system still remains valid. (The area, A, is to be left unaltered in this term as well as other terms, and the lengths to be left unaltered in the last term.)

3. THE MODESUPERPOSITION TECHNIQUE FOR TREATING THE TIMEDERIVATIVE

In this section we develop the modesuperposition technique for treating the timederivative when solving timedependent problems with the Galerkin method, i.e. when solving the problem:

$$C\frac{dh}{dt} = -Bh + b(t) \quad , \quad h(t_0) = \hat{h}$$

(cf. equation (10)). The basic idea of using such a technique is not new. It has f.ex. been proposed for a more general class of problems by Wait and Mitchell (1971) (cf. also Mitchell and Wait (1977)). Independently Kjaran (1976) proposed its use for solving groundwater flow problems in particular. The development below is an extension of that work, where we pay special attention to computationally efficient implementations of the technique.

For theoretical, rather than computational, reasons we shall find it convenient to divide $h(t)$, the approximate groundwaterlevel at the gridpoints, into two parts:

$$h(t) = h_o + h_1(t)$$

where h_o, the stationary part, satisfies the linear equation

$$Bh_o = \bar{b} \qquad (16)$$

and h_1, the transient part, satisfies the linear differential equation:

$$C\frac{dh_1}{dt} = -Bh_1 + (b(t) - \bar{b}) \quad , \quad h_1(t_o) = \hat{h} - h_o \qquad (17)$$

It is then understood that the vector $\bar{b}$ includes the contributions from the timeindependent nonhomogenous boundary conditions, i.e. the last two terms in equation (13) and possibly also some contribution from a long-term averagevalue for the source

function R.

Let λ_i and $\underset{\sim}{\phi}_i$, i = 1,..,N, denote the eigenvalues and corresponding eigenvectors of the eigenvalue problem:

$$\tilde{B}\underset{\sim}{\phi} = \lambda \tilde{C}\underset{\sim}{\phi}$$

where $$\tilde{B} = \frac{1}{T_o} B \quad , \quad \tilde{C} = \frac{1}{AS_o} C,$$

i.e. we have normalized B with respect to some reference transmissivity, T_o, and normalized C with respect to some reference storage coefficient, S_o, and the area A. We furthermore assume that the eigenvectors are normalized such that $\underset{\sim}{\phi}_i^T \tilde{C} \underset{\sim}{\phi}_i = 1$ and the eigenvalues are ordered such that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$. By writing

$$\underset{\sim}{h}_1(t) = \sum_{i=1}^{N} h_i(t)\underset{\sim}{\phi}_i \quad , \quad \underset{\sim}{b}(t) - \underset{\sim}{\bar{b}} = \sum_{i=1}^{N} b_i(t)\, \tilde{C}\underset{\sim}{\phi}_i ,$$

and substituting these expressions into equation (17) we then obtain that

$$\frac{dh_i}{dt} = -\frac{1}{K_i} h_i + \frac{1}{AS_o} \underset{\sim}{\phi}_i^T (\underset{\sim}{b}(t)-\underset{\sim}{\bar{b}}) \qquad h_i(t_o) = \underset{\sim}{\phi}_i^T \tilde{C}(\underset{\sim}{\hat{h}}-\underset{\sim}{h}_o) \quad (18)$$

where $$K_i = \frac{AS_o}{\lambda_i T_o} \qquad i = 1,2,..,N \quad (19)$$

and hence that:

$$\underset{\sim}{h}_1(t) = \sum_{i=1}^{N} (\underset{\sim}{\phi}_i^T \tilde{C}(\underset{\sim}{\hat{h}}-\underset{\sim}{h}_o))\, e^{-\frac{t-t_o}{K_i}}\, \underset{\sim}{\phi}_i$$

$$+ \frac{1}{AS_o} \sum_{i=1}^{N} \left(\int_o^{t-t_o} \underset{\sim}{\phi}_i^T (\underset{\sim}{b}(t-\tau)-\underset{\sim}{\bar{b}}) e^{-\frac{\tau}{K_i}} d\tau \right) \underset{\sim}{\phi}_i \quad (20)$$

which is our basic solution form for the transient part. If we wish to include the stationary part we can of course do so, simply by setting $\underset{\sim}{h}_o = \underset{\sim}{\bar{b}} = \underset{\sim}{o}$ on the RHS of equation (20).

Before considering the computational aspects of the solution form we observe that equation (18) is a differential equation for a linear reservoir with a timeconstant K_i. The response of the approximate groundwaterlevel at each gridpoint to the infiltration or pumping within the region is therfore exactly

analogous to the outflow response of a system of N parallel simple linear resevoirs as illustrated in figure 4, the inflow to the reservoir associated with timeconstant K_i being determined by the eigenvector $\underset{\sim}{\phi}_i$. This interpretation of the groundwater response is often very useful from a hydrological standpoint.

By comparison we have that the transient part of the theoretical solution to equation (1) to (5), i.e. the function $h_1(x,y,t)$ satisfying,

$$S\frac{\partial h_1}{\partial t} = Lh_1 + (R-\bar{R}) \quad \text{in } \Omega$$

$$h_1 = 0 \quad \text{on } \partial\Omega_1$$

$$-Nh_1 = 0 \quad \text{on } \partial\Omega_2$$

$$h_1(x,y,t_o) = \hat{h}(x,y) - h_o(x,y) \quad \text{in } \Omega$$

where $\bar{R}(x,y)$ denotes long-term average for the source function and $h_o(x,y)$ is the stationary part satisfying

$$Lh_o = -\bar{R} \quad \text{in } \Omega$$

$$h_o = g \quad \text{on } \partial\Omega_1$$

$$-Nh_o = f \quad \text{on } \partial\Omega_2 \quad ,$$

may be expressed as:

$$h_1(x,y,t) = \frac{1}{AS_o}\sum_{i=1}^{\infty}\left[\iint_{\Omega}\phi_i(\xi,\eta)\, S(\xi,\eta)(\hat{h}(\xi,\eta)-h_o(\xi,\eta))d\xi d\eta\right]e^{-(\frac{t-t_o}{K_i})}\phi_i(x,y)$$

$$+\frac{1}{AS_o}\sum_{i=1}^{\infty}\left[\int_{o}^{t-t_o}\left[\iint_{\Omega}\phi_i(\xi,\eta)(R(\xi,\eta,\tau)-\bar{R}(\xi,\eta))\, d\xi d\eta\right]e^{-\frac{\tau}{K_i}}d\tau\phi_i(x,y)\right]$$

Here K_i is defined as in equation (19) whereas λ_i and $\phi_i(x,y)$, $i = 1,2,..$ are now the eigenvalues and eigenfunctions of the eigenproblem

$$\frac{1}{T_o}L\phi + \lambda\frac{1}{AS_o}S\phi = 0 \quad \text{in } \Omega$$

$$\phi = 0 \qquad \text{on } \partial\Omega_1$$

$$-N\phi = 0 \qquad \text{on } \partial\Omega_2$$

and the eigenfunctions are normalized so that

$$\frac{1}{AS_o} \iint_{\Omega} S(\xi,\eta)\phi_i^2(\xi,\eta)d\xi d\eta = 1$$

It has been observed by Kjaran (1976) that the response of the transient part of the exact theoretical groundwater level at any point within the region to infiltration or pumping is again exactly analogous to the outflow response of a system of parallel linear reservoirs, the only difference being that the reservoirs in this case will be infinitely many. He further observed that the response of the total outflow out of the region can be viewed in exactly the same way. Although these observations were restricted to the case of isotropic flow the results are readily extended to the case of anisotropic flow.

Turning to the computational aspects of the solution form in equation (20), this has to be contrasted with the computational effort required in a finite difference method. A brief account of these methods can be found f.ex. in Mitchell and Wait (1977) and Pinder and Gray (1977). The relevant facts for the present discussion are that when advancing each timestep, Δt, in the finite difference method the main computational task is that of solving a linear system of the type

$$\left[aC + \frac{1}{\Delta t}B\right]\underset{\sim}{h} = \underset{\sim}{c}$$

where C and B are the NxN matrices in equation (17) and the vector $\underset{\sim}{c}$ can be calculated explicitly. a is a parameter depending on the choice of method. a = 0 corresponds to an explicit method. The advantage of such a method is that after we have obtained triangular factors for B at the start of the computation we only have to perform a forward and a backward substitution at each timestep regardless of whether we change the timestep, Δt. Similarily, we only have to perform forward and backward substitutions if we recalculate $\underset{\sim}{h}$ for a new input function. However, stability considerations in general restrict the maximum timestep to such a small value

that we opt for an implicit method with a > 0. For such a method, on the other hand, we have to recalculate the triangular factors of $aC+\frac{1}{\Delta t}B$ each time that we change the timestep. Moreover, if we wish to recalculate $\underset{\sim}{h}$ for new input functions, we have to choose between carrying out these calculations simultaneously, retaining the triangular factors from each stepchange, or recalculating these factors.

Returning to equation (20) we first note that the convolution integral can in general be conveniently dealt with if we make use of the fact that

$$\int_0^{t+\Delta t-t_o} f(t+\Delta t-\tau)e^{-\frac{\tau}{K_i}}\,d\tau = e^{-\frac{\Delta t}{K_i}}\int_0^{t-t_o} f(t-\tau)e^{-\frac{\tau}{K_i}}\,d\tau + \int_0^{\Delta t} f(t+\Delta t-\tau)e^{-\frac{\tau}{K_i}}\,d\tau$$

when we advance from time t to time t+Δt. Secondly, we note that if we want to calculate $\underset{\sim}{h}(t)$ for a new input function (infiltration, pumping), the eigenvectors and eigenvalues remain the same so we just have to reevaluate the convolution integral to get a new solution, irrespectively of whether we use (variable timesteps) when advancing the integral value in equation (20).

A similar situation arises if we have a leaky aquifer with the leakage proportional to the drawdown. Then the eigenproblem does not have to be solved again if we introduce a constant leakage factor γ, since the eigenvectors will remain unaltered and the new eigenvalues are simply given by

$$\lambda_{new} = \lambda + \frac{A\gamma}{T_o} \quad .$$

For large N, however, the number of convolution integrals to be calculated would in general be too great to make the solution form in equation (20) computationally attractive in comparison with finite difference methods, but this picture changes if acceptable approximations to $\underset{\sim}{h}(t)$ can be obtained with the knowledge of only a few, M say, of the smallest eigenvalues (largest timeconstants) and the associated eigenvectors. Furthermore, in that case, the computational effort involved in obtaining the eigenvalues and eigenvectors which would be very significant for large N can be cut considerably. Powerful algorithms have been developed for these type of problems e.g

the socalled subspace iteration algorithm. The algorithm, which can be viewed as an extension of the well known inverse iteration algorithm, obtains the M smallest eigenvalues and associated eigenvectors simultaneously and involves only at each iterationstep a solution of a reduced eigenproblem of size K, as well as K linear systems of the form $B\underset{\sim}{x} = \underset{\sim}{y}$ where $K \approx M+8$ and B is the NxN matrix in equation (17). The convergence is in general very rapid. It has been used sucessfully for solving structural problems (see Bathe et al. (1974) and Strang and Fix (1973)).

In order to see when this type of approach is possible it is convenient to assume that

$$\underset{\sim}{b}(t) = \underset{\sim}{b}_o \text{ (constant) in the interval } [\hat{t}-\Delta t, \hat{t}] \equiv [t_o, \hat{\ }]$$

The solution form in equation (20) can then be rewritten as follows:

$$\underset{\sim}{h}(\hat{t}) = \sum_{i=1}^{N} (\underset{\sim}{\phi}_i^T \tilde{\underset{\sim}{C}} \hat{\underset{\sim}{h}}) e^{-\frac{(t-t_o)}{K_i}} \underset{\sim}{\phi}_i + \frac{1}{AS_o} \sum_{i=1}^{N} \left(\int_{\Delta t}^{t-t_o} \underset{\sim}{\phi}_i^T \underset{\sim}{b}(\hat{t}-\tau) e^{-\frac{\tau}{K_i}} d\tau \right) \underset{\sim}{\phi}_i$$

$$+ B^{-1}\underset{\sim}{b}_o - \frac{1}{AS_o} \sum_{i=1}^{N} (\underset{\sim}{\phi}_i^T \underset{\sim}{b}_o) K_i e^{-\frac{\Delta t}{K_i}} \underset{\sim}{\phi}_i \qquad (21)$$

where we have used the fact that

$$\frac{1}{AS_o} \sum_{i=1}^{N} (\underset{\sim}{\phi}_i^T \underset{\sim}{b}_o) K_i \underset{\sim}{\phi}_i = B^{-1}\underset{\sim}{b}_o$$

and chosen to combine again $\underset{\sim}{h}_o$ and $\underset{\sim}{h}_1(t)$.
It then follows that provided Δt is sufficiently large such that

$$K_{M+1} e^{-\frac{\Delta t}{K_{M+1}}} < \varepsilon \qquad (22)$$

where ε denotes some error tolerance that will depend mainly on the size of $\underset{\sim}{b}(t)$ we can expect to obtain a satisfactory approximation for $\underset{\sim}{h}(\hat{t})$ even if we replace N by M on the RHS of equation (21). Note, however, that this is not the same as replacing N by M in equation (20), since all the eigenvectors are

implicitly included in the term $B^{-1}\underset{\sim}{b}_o$ although we do not require explicit knowledge of the vectors $\underset{\sim}{\phi}_i$, $i \geq M+1,..,N$. Rather, this approximation amounts to replacing all the timeconstants K_i, $i \geq M+1,..,N$, by zero or in terms of the parallel reservoirs analogy we have treated the reservois M+1 to N with no timedelay.

In order to illustrate this further we present in figure 5 the results of a small example, where we take the flow area to be a square with an area, A, of $10^8 m^2$, and the source term to be given in form of constant pumping, Q, of $1 m^3/s$ at the centre of the square. We let the aquifer be isotropic and homogenous with a transmissivty coefficient, T, of $0.2 m^2/s$ and a storage coefficient, S, of 0.06 and prescribe constant head on the boundary. We have then calculated the drawdown at an observation well halfway between the centre and the boundary as a function of time. (On the axes in the figure we show a dimensionless drawdown, $s^1 = \frac{Ts}{Q}$, and a dimensionless time, $t^1 = \frac{Tt}{AS}$, as well as an actual drawdown and time; the results may thus be interpreted for different values of A, Q, T, and S). This example has been considered by Kjaran (1976) who shows that using a uniform triangular grid with 48 internal grid points in the Galerkin finite element method we get a solution, referred to as a 48 eigenfunctions solution in the figure, which is in close agreement with the theoretical solution. Using the same number of internal points, but including only the first term in the sums in equation (20), we get the solution referred to as a 1 eigenfunction solution, whereas if we include only the first term in the sums in equation (21), as well as the term $B^{-1}\underset{\sim}{b}_o$, we get the solution referred to as a corrected 1 eigenfunction solution. As can be seen the latter approximation is an extremely good one except for very small timevalues, whereas the first one is relatively poor (in fact almost as poor as a 9 eigenfunction solution based on 9 internal points).

In practice when $\underset{\sim}{b}(t)$ is in fact often a piecewise-constant vectorfunction we find that we are justified in using equation (21) with N (or rather M) as low as 1-10 for many t-values of interest. It is of interest to observe, however, that when equation (22) is only satisfied for large values of M, for some given time, $\hat{t}$, that rather than computing additional eigenvalues and eigenvectors we have either of the following two options:

I) We can increase Δt beyond the range where the approximation $\underset{\sim}{b}(t) \approx \underset{\sim}{b}_o$ is valid, provided we add to the RHS of equation (21) a correction term which is the solution of the differential equation

$$C\frac{d\underset{\sim}{h}}{dt} = -B\underset{\sim}{h} + \underset{\sim}{b}(t) - \underset{\sim}{b}_o \qquad \underset{\sim}{h}(\hat{t}-\Delta t) = 0$$

at the time $\hat{t}$. This solution can in turn be obtained by a finite difference method, and this may be more economical than using such a method over the whole interval, if the difference between the t-values where we require a solution is considerably larger than Δt.

II) We can choose the interval $[\hat{t}-\Delta t, t]$ in such a way that $\underset{\sim}{b}(t)$ can be replaced by a low order polynomial on the interval rather than a constant. If we assume f.ex. that $\underset{\sim}{b}(t) = \underset{\sim}{b}_o + (t-\hat{t})\underset{\sim}{b}_1 + (t-\hat{t})^2\underset{\sim}{b}_2$ in the interval $[\hat{t}-\Delta t, \hat{t}]$ then the RHS of equation (21) remains valid provided the last two terms are replaced by the following:

$$B^{-1}(\underset{\sim}{b}_o - B^{-1}(\underset{\sim}{b}_1 - B^{-1}2\underset{\sim}{b}_2)) - \frac{1}{AS_o}\sum_{i=1}^{N}[(\underset{\sim}{\phi}_i^T\underset{\sim}{b}_o)$$

$$-(\underset{\sim}{\phi}_i^T\underset{\sim}{b}_1)(K_i + \Delta t) + (\underset{\sim}{\phi}_i^T\underset{\sim}{b}_2)(2K_i^2 + 2\Delta tK_i + \Delta t^2)]K_i e^{-\frac{\Delta t}{K_i}}\underset{\sim}{\phi}_i \quad (23)$$

Although these terms are somewhat more complex, the increase in computational effort will in many cases be more than offset by the possible decrease in the number of eigenfunctions, M. We further note that provided the triangular factors for the matrix B are available we only need to perform three forward and backward substitutions in order to evaluate the first term in (23) compared with one forward and one backward substitution in order to evaluate $B^{-1}\underset{\sim}{b}_o$ in equation (21).

Finally, we observe that the triangular factors for B (Cholesky factors since B is symmetric and positive-definite) required for the evaluation of $B^{-1}\underset{\sim}{b}_o$ in equation (21) or the corresponding term in equation (23) at each t-value of interest can be obtained once and for all at the start of the computation, as is the case for an explicit finite difference method, and they are also precisely the triangular factors required in the

subspace iteration used to calculate the eigenvalues and eigenvectors (cf. the description of that algorithm above).

Summarizing what we consider the most important aspects of the mode superposition technique, we have that:

1) It allows us to interpret the response of the approximate groundwaterlevel to infiltration or pumping in terms of simple parallel linear reservoirs, an interpretation that carries over from the exact theoretical groundwaterlevel.
2) It may offer a computationally attractive alternative to the use of finite difference methods when one of the following conditions are satisfied:

a) The input function, $\underset{\sim}{b}(t)$, can be approximated by a constant (or a low-order polynomial) over relatively large subintervals of the total time interval of interest.
b) Groundwaterlevel values are only required at relatively few t-values compared with the total time interval of interest.
c) The groundwaterlevel, $\underset{\sim}{h}(t)$, has to be calculated for various input values (or possibly various leakage factors), but the same values for transmissivity and storage coefficients.

4. CALCULATION OF FLOW ACROSS BOUNDARIES

In order to simplify the presentation we restrict our attention in this section to the steady state problem, where the LHS of equation (1) is replaced by zero, unless otherwise specified. We assume that we have determined the level values, h_i, at all nodal points and that we are now interested in determining the flow distribution across internal as well as external boundaries e.g boundaries A1-A9, B1-B7, and C1-C3 in figure 6.

The most straightforward approach would be to obtain the flow values from the normal gradients of the approximate solution, $\bar{h}$, within the triangular elements next to the boundaries. In the isotropic case this amounts to using the following formula for obtaining the total flow across the edge ij in figure 7:

$$\frac{1}{2}l_{ij}(f_{i+} + f_{j-}) = \frac{T}{4A}\left[2l_{ij}^2 h_k + (l_{jk}^2 - l_{ki}^2 - l_{ij}^2)h_j + (l_{ki}^2 - l_{ij}^2 - l_{jk}^2)h_i\right] \qquad (24)$$

where A denotes the area of the element ijk, l_{ij} the length of the edge between points i and j and T the constant transmissivity within the element. The same formula remains valid in the anisotropic case if we transform the (x,y)-coordinates according to equation (15) and recalculate the lengths of the edges in the new coordinate system as well as replacing T by $\lambda_1 \cdot \lambda_2$ (cf. comments before and after equation (15)).

The above formula, however, has the following disadvantages:

i) When flow values are calculated across a closed boundary they will in general not match the infiltration within the boundary, indeed discrepancies of up to 50% are not uncommon in practical problems.

ii) Flow values based on it will not agree with specified flow values on the boundary $\partial\Omega_2$.

iii) At an internal boundary we obtain different flow values depending on, on which side the triangular element is that we base our calculations on.

In order to overcome these disadvantages we propose that use is made of the following relationship, if we are interested in the flow across boundary mij in figure 7:

$$\frac{1}{6}[l_{mi}(f_{m+} + 2f_{i-}) + l_{ij}(2f_{i+} + f_{j-})]$$

$$= -\sum_{\Delta} \frac{T}{8A}[2l_{jk}^2 h_i + (l_{ij}^2 - l_{jk}^2 - l_{ki}^2)h_j \quad (25)$$

$$+(l_{ki}^2 - l_{ij}^2 - l_{jk}^2)h_k] + \frac{A}{12}[2R_i + R_j + R_k] - \alpha_i Q .$$

The summation sign on the RHS denotes that we are to add to the contribution from the triangular element ijk, presented on the RHS, analagous contributions from the triangular elements ikl and ilm. The relationship has been obtained by combining those equations in the element contributions given in equation (14) that are associated with node i and by letting flow terms across common boundaries cancel out just as is done in the assembly process described earlier. We shall refer to the relationship as the Galerkin flow formula although it might also be referred to as a generalized flow formula (cf. the terminology of Larock and Herrman (1976)). We have assumed

that we have a pumping Q at the point p and define α_i as in equation (14). Since we are restricting ourselves to the steady state problem the LHS in equation (14) has been set to zero, but the only change that has to be made in the Galerkin flow formula if we drop this restriction is to replace R_i by $R_i - S\frac{dh_i}{dt}$. Anisotropy is dealt with in exactly the same way as described before and after equation (15). Finally it may be noted that the first term on the RHS of the flow formula (25), after we have summed over the appropriate triangles, is exactly half the gradient inflow through the boundary jklm (cf. equation (24)).

The Galerkin flow formula may be implemented in two different ways:

A. We may interpret the LHS as giving the total flow across the boundary from the halfway point between nodes m and i to the halfway point between nodes i and j.

B. We may combine flow formulae for all the nodes along the total boundary of interest into a tridiagonal linear system and calculate from it separate f values.

The second implementation is most readily described by considering the simple example in figure 8. Here the boundary between points 1 and 5 may be thought of as either an external or an internal boundary (cf. figure 6). We assume that the values f_{0+}, and f_{6-} are known from prescribed boundary conditions. We then go on to set 1) $f_{2-} = f_{2+}$, 2) $f_{3+} = f_{4-}$, and 3) $f_{4-} = f_{4+}$. 1) is of course exact since the boundary is straight around node 2. 2) is an approximation that remains reasonable as long as the edge between nodes 3 and 4 is relatively short; we shall refer to it as a short edge approximation. 3) is an approximation that remains reasonable as long as the angle made by the boundary at node 4 is relatively flat; we shall refer to it as a flat angle approximation. The Galerkin flow formulae for nodes 1 to 5 can now be combined into the following linear system:

$$\begin{bmatrix} \frac{l_{12}}{3} & \frac{l_{12}}{6} & 0 & 0 & 0 \\ \frac{l_{12}}{6} & \frac{l_{12}+l_{23}}{3} & \frac{l_{23}}{6} & 0 & 0 \\ 0 & \frac{l_{23}}{6} & \frac{l_{23}}{3} & \frac{l_{24}}{2} & 0 \\ 0 & 0 & 0 & \frac{3l_{34}+2l_{45}}{6} & \frac{l_{45}}{6} \\ 0 & 0 & 0 & \frac{l_{45}}{6} & \frac{l_{45}}{3} \end{bmatrix} \begin{bmatrix} f_{1+} \\ f_{2+} \\ f_{3-} \\ f_{4+} \\ f_{5-} \end{bmatrix} = \begin{bmatrix} b_1 - \frac{l_{01}(2f_{1-}+f_{0+})}{6} \\ b_2 \\ b_3 \\ b_4 \\ b_5 - \frac{l_{56}(2f_{5+}+f_{6-})}{6} \end{bmatrix}$$

where b_i denotes the contribution from the RHS in equation (25) at node i. This tridiagonal system can then be solved to obtain values for f_{1+}, $f_{2-} = f_{2+}$, f_{3-}, $f_{3+} = f_{4-} = f_{4+}$, and f_{5-}. We observe that if the boundary between nodes 1 and 5 had been straight, no approximation would be needed (we could then set $f_{3+} = f_{3-}$ rather than setting $f_{3+} = f_{4-}$). We also observe that if f_{1-} and f_{0+} f.ex were not known this could be counteracted by using a short edge approximation between nodes 1 and 2 (i.e. by setting $f_{1+} = f_{2-}$).

Irrespectively of whether we make use of implementation A or B it is a direct consequence of the fact that the Galerkin flow formula is consistent with the assembly process in the Galerkin finite element method that we will get exact match between calculated flow on one hand and prescribed inflow and infiltration on the other (cf. the disadvantages (i) and (ii) described above for formula (24)). Indeed, if we have obtained the final assembled system in equation (10) we can read the Galerkin flow formula from those equations that are associated with external boundary nodes, where flow has been prescribed. Secondly, when calculating flow across an internal boundary it will not matter on which side the triangles are, on which the Galerkin flow formula is based (cf. the disadvantage (iii) described above).

In order to illustrate the Galerkin flow formule and its two implementations we present in figure 9 the results of calculated flow values across the internal boundary A1-A9 in the region in figure 6. For this region zero flow is specified across the shaded part of the boundary and a prescribed groundwaterlevel on the remaining part of it. The calculated level values within

the region vary from 280 to 560 m, transmissivty values from $1.26 \cdot 10^{-4}$ to $1.26 \cdot 10^{-1}$ m^2/s and infiltration from 500 to 1050 mm/year. The arrows indicate approximate flow lines.

Downstream and upstream gradient flow values in figure 9 stand for flow values based on equation (24) and using elements above and below the boundary respectively. Galerkin flow values stand for flow values based on equation (25). The horizontal line segments correspond to inplementation A. Here, as well as for the down-and upstream gradient flow values, we have divided the total flow across the appropiate edge with the length of the edge in order to obtain the given values. The almost continous line corresponds to implementation B. We have made a flat angle approximation at points A2, A3, A4, A5, and A7 and a short edge approximation along edges A5-A6 and A8-A9. For both implementations we make use of the fact that we have prescribed zero flow across the edges A0-A1 and A9-A10 or A0'-A1 and A9-A10' depending on whether we calculate the flow values from elements above or below the boundary respectively (the results, as already mentioned, will be identical). The total flow across A1-A9 according to the Galerkin formulae is 3.94 m^3/s, and this matches exactly the infiltration above the boundary. It is interesting to note that downstream gradient flow values which lead to almost correct total flow match the Galerkin flow values much worse than the upstream gradient flow values.

The difference between the total outflow through the boundary B1-B9 and the total inflow through the boundary C1-C3 in the same example is 8.93 m^3/s according to the Galerkin flow formula, matching the infiltration within the whole region. An interesting feature here is that, although the corresponding difference is only 6% higher if we calculate upstream gradient flow across B1-B8 and downstream gradient flow across C1-C3, the Galerkin flow formule gives in fact ca. 30% greater outflow through B1-B8 and almost three times as great inflow through C1-C3.

It is of interest to note the form that the RHS of equation 25 takes in the regular case shown on the left in figure 10, where $l_{ij} = l_{il} = l_{im} = l$, say, and we further assume that we have the same transmissivity, T, within all the triangles and omit the pumping term. The RHS becomes:

$$-T[2h_i - \frac{1}{2}(h_j + h_m) - h_l] + \frac{l^2}{24}[6R_i + R_j + R_k + 2(R_m + R_l)] \quad (26)$$

By applying Taylor expansions round the point i it can be shown that this is in fact a second order difference approximation to $-lT \left.\frac{\partial h}{\partial x}\right|_i$ for a function,h,that satisfies the differential equation

$$-\Delta(T\Delta h) = R$$

and indeed a third order difference approximation to

$$-lT \frac{1}{6}\left[\left.\frac{\partial h}{\partial x}\right|_m + 4\left.\frac{\partial h}{\partial x}\right|_i + \left.\frac{\partial h}{\partial x}\right|_j\right]$$

if $\frac{\partial R}{\partial y} = 0$. The difference approximation $-l(\frac{h_i - h_l}{l})$, corresponding to the calculated groundwatergradient is of course only a first order approximation. However, it should be mentioned that in the regular case shown on the right in figure 10 the RHS of formula (25) becomes identical to (26) except the term R_k is missing which in turn means that the improved order of the difference approximation is lost unless R = 0.

The fact that the Galerkin finite element method may lead to high order difference approximation on the boundary is observed by e.g Strang and Fix (1973 ,p.33) in the case of ordinary differential equations. More generally, the fact that we can obtain through the Galerkin method accurate approximations for the flow values at particular points of the domain in onedimensional field problems, using an approach closely related to the one presented here for twodimensional problems, was observed by J.A. Wheeler (1973) and subsequently analysed by M.F. Wheeler (1974) and more recently by Dupont (1976). Their analysis does not seem to be readily extendable to twodimensional problems. In particular it should be noted that the implementation of the Galerkin flow formula in the onedimensional case does not pose any added problems since only one f value will enter into the analoge of formula (25).

Larock and Herrmann (1976) have proposed a scheme for obtaining accurate flow values at the cornerpoints of 4 node isoparametric quadrilateral elements when solving general field problems. They make use of generalized flow relations

implicit in equation (14)(or rather its rectangular element counterpart) along with some additional information. They restrict, however, their attention to information obtained from a single element rather than combining element contributions as is done in formula (25).

Rodi (1976) has developed a method for calculating accurate gradients in electromagnetic field problems and demonstrated its usefulness. Although his motivation and derivation differs from ours and 4 point rectangular elements are used in the solution, the basic formula of his method can be shown to be analogus to the Galerkin flow formula (25) combined with implementation B, on a straight boundary.

Finally, the second author has used Galerkin method with rectangular elements for solving linear heat conduction- convection problems in a rectangular region and calculated heat flow with the same type of Galerkin flow formula. In this case the accuracy of the flow values can be studied by reducing the size of the elements and employing extrapolation methods. It has been observed that the order of accuracy of the flow values becomes approximately 2 rather that approximately 1 as is the case with values based on simple gradient approximations analogus to those in formula (24). However, the benefit of using the Galerkin flow formula seems to diminish as the convective term becomes more dominant, a result that may relate to the fact that the Galerkin method itself is known to present problems when dealing with this type of problems, cf. Mitchell and Wait (1977, p. 184).

Summarizing what we consider the most important aspects of the Galerkin flow formule we have that:

1) It guarantees complete match with prescribed infiltration and inflow.
2) It corresponds to improved order difference approximations for the flow in a special regular case arising when we have a uniform triangularization of a rectangular region.
3) Its counterpart in onedimensional field problems has been analysed and proved to result in an improved accuracy, and there is experimental evidence of the usefulness of its counterpart in twodimensional field problems with regular rectangular elements and straight boundaries.

REFERENCES

Bathe, K.J., Wilson, E.L., Peterson, F.E. (1973). SAP IV. A Structural Analysis Program For Static and Dynamic Response of Linear Systems. College of Engineering. University of California. Report No EEPC 73-11.

Dupont T. (1976). A Unified Theory of Superconvergence for Galerkin Methods For Two-Point Boundary Problems. Siam. J. Numer. Anal., 13, 3: 362-368.

Kjaran S.P. (1976). Theoretical and Numerical Models of Groundwater Reservoir Mechanism. Institute of Hydrodynamics and Hydraulic Engineering, Technical University of Denmark, Series Paper no. 13.

Kjaran, S.P., Sigurdsson, S.T. (1978). Two Techniques Associated with the Galerkin Method for Solving Groundwater Flow Problems. Finite Elements in Water Resources 2: 1.227-1.247. Pentech Press.

Larock, B.E., Herrman, L.R. (1976). Improved Flux Prediction Using Low Order Finite Elements. Finite Elements in Water Resources 1: 1.103-1.114.

Mitchell, A.R., Wait, R. (1977). The Finite Element Method in Partial Differential Equations. John Wiley and Sons.

Neuman, S.P., Whitherspoon, P.A. (1970). Finite Element Method of Analysing Steady Seepage with a Free Surface. Water Resources Research, 6, 3: 889-897.

Pinder, G.F., Friend, E.O. (1972). Application of Galerkin's Procedure to Aquifer Analysis. Water Resources Research, 8, 1: 108-120.

Pinder, G.F., Gray, W.G. (1974). Galerkin Approximation of the Time Derivative in the Finite Element Analysis of Groundwater Flow. Water Resources Research, 10, 4: 821-828.

Pinder G.F., Gray, W.G. (1977). Finite Element Simulation in Surface and Subsurface Hydrology. Academic Press.

Rodi, W.L. (1976). A Technique for Improving the Accuracy of Finite Element Solutions for Magnetotelluric Data. Geophysical Journal of the R.A.S. 44, 2: 483-506.

Strang, G., Fix, G.J.(1973). An Analysis of the Finite Element Method. Prentice-Hall, Inc.

Wait, R., Mitchell, A.R. (1971). The Solution of Time Dependent Problems by Galerkin Methods. J. Inst. Maths. Applics. 7: 241-250.

Wheeler, J.A. (1973). Simulation of Heat Transfer from a Warm Pipeline Buried in Permafrost. 74th Nat. Meeting Amer. Inst. of Chem. Eng. New Orleans, March 1973.

Wheeler, M.(1974). A Galerkin Procedure For Estimating the Flux for Two-Point Boundary Value Problems. Siam. J. Num. Anal. 11, 4: 764-768.

Zienkiewicz, O.C. (1971). The Finite Element Method in Engineering Science. McGraw-Hill.

ÁGRIP (ABSTRACT IN ICELANDIC)

Fjallað er um tvö vandamál, sem upp koma, þegar búta-aðferð Galerkins er beitt til þess að ákvarða nálgunarlausn á línulegri parabólskri diffurlíkingu fyrir tímaháðar grunnvatns-breytingar í tvívíðum, misleitnum og ósamkynja vatnsleiðara.

Fyrra vandamálið snertir meðferð á tímaháða þætti grunn-vatnshæðarinnar. Bent er á, að í stað þess að ákvarða tíma-breytingar með aðferð endanlegra mismuna, eins og algengt er, komi til álita að nota eiginlausnir. Er þá um að ræða nálgunar-eiginlausnir, sem ákvarðaðar eru út frá Galerkinnálgun diffur-líkingarinnar. Sýnt er í fyrsta lagi, að út frá eiginlausnunum má líta á grunnvatnsgeyminn sem samsíða línulegar miðlanir. Slíkt er kostur við vatnafræðilega túlkun. Í örðu lagi er gerð gerin fyrir því, að nægilegt er að þekkja aðeins eiginlausnir, sem svara til nokkurra minnstu eigingildanna, til þess að ákvarða grunnvatnshæðir með viðhlítandi nákvæmni, í því tilviki til dæmis, þegar írennsli er fasti á hlutfallslega stórum tíma-hlutbilum. Sé skynsamlega að reikningum staðið, getur eigin-lausnanálgun þá jafnframt orðið reiknislega hagkvæmari heldur en mismunanálgun.

Síðara vandamálið snertir ákvörðun á grunnvatnsrennsli gegnum ytri og innri jaðra út frá reiknuðum grunnvatnshæðum. Vitað er að rennslisgildi, sem ákvörðuð er beint út frá reiknuðum grunnvatnshalla, geta orðið mjög ónákvæm, enda þótt grunnvatns-hæðirnar sjálfar hafi verið ákvarðaðar með viðhlítandi nákvæmni. Sýnt er, að með því að nota svokölluð alhæfð rennslissambönd, sem fólgin eru í Galerkinnálgun diffurlíkingarinnar, má a.m.k. tryggja samsvörun milli reiknaðs útrennslis og gefins innrennslis. Fjallað er um, hvernig nota má þessi sambönd á reglulegu svæði, sem skipt hefur verið í þríhyrningsbúta. Loks er gerð grein fyrir því á grundvelli einfaldaðra dæma, hvers vegna ætla má, að rennslisgildi, sem eru ákvörðuð út frá alhæfðum rennslis-samböndum, séu að jafnaði nákvæmari en gildi ákvörðuð út frá grunnvatnshalla.

M Y N D I R

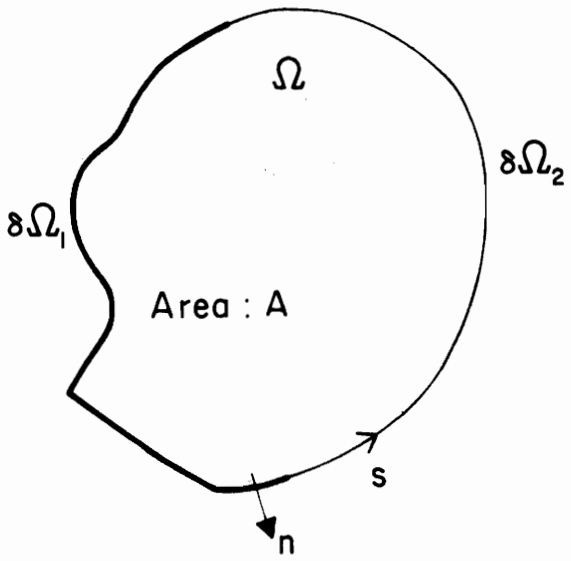

Figure 1. Schematic groundwater area

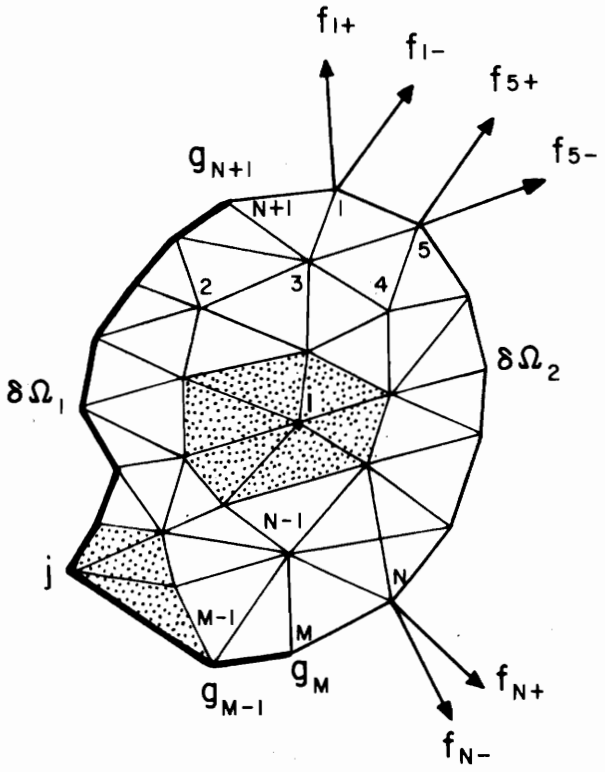

Figure 2. Triangularization of groundwater area

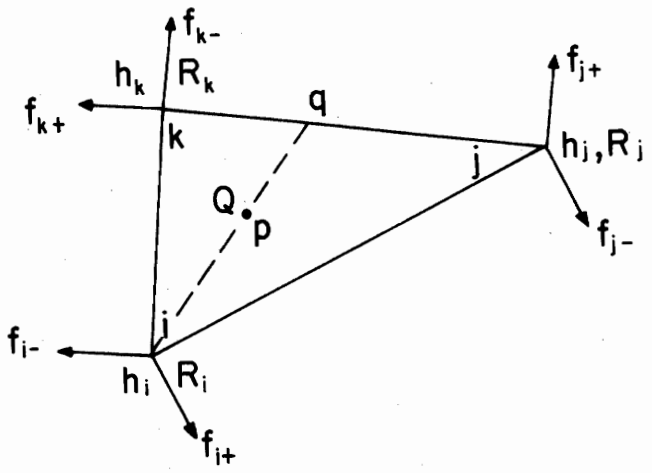

Figure 3. Schematic element

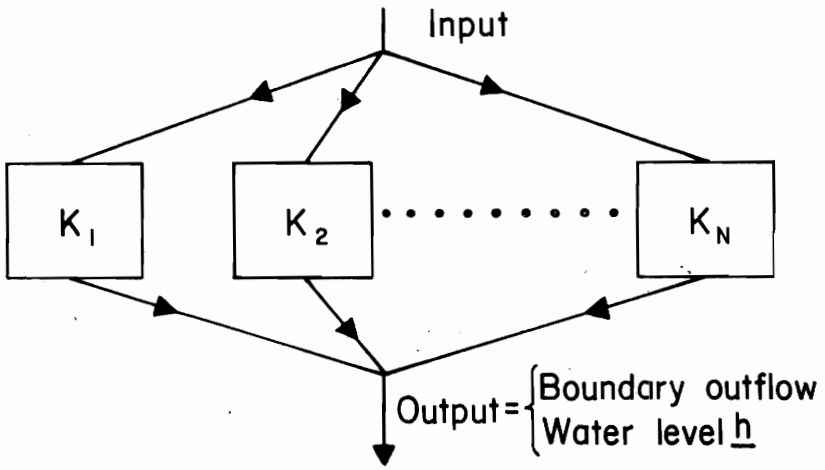

Figure 4. Linear reservoir model

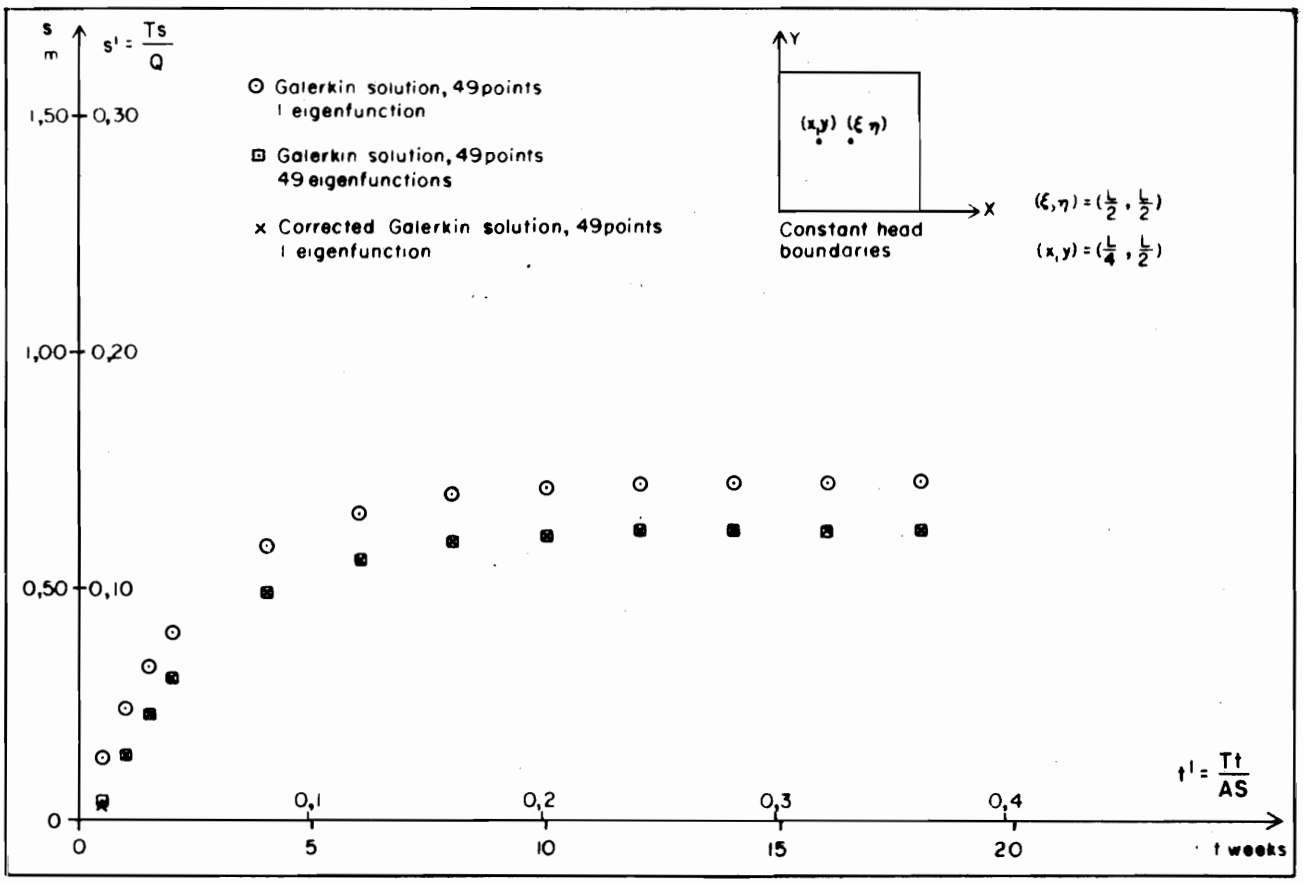

Figure 5. Drawdown in a square area vs. time. Comparison of solutions.

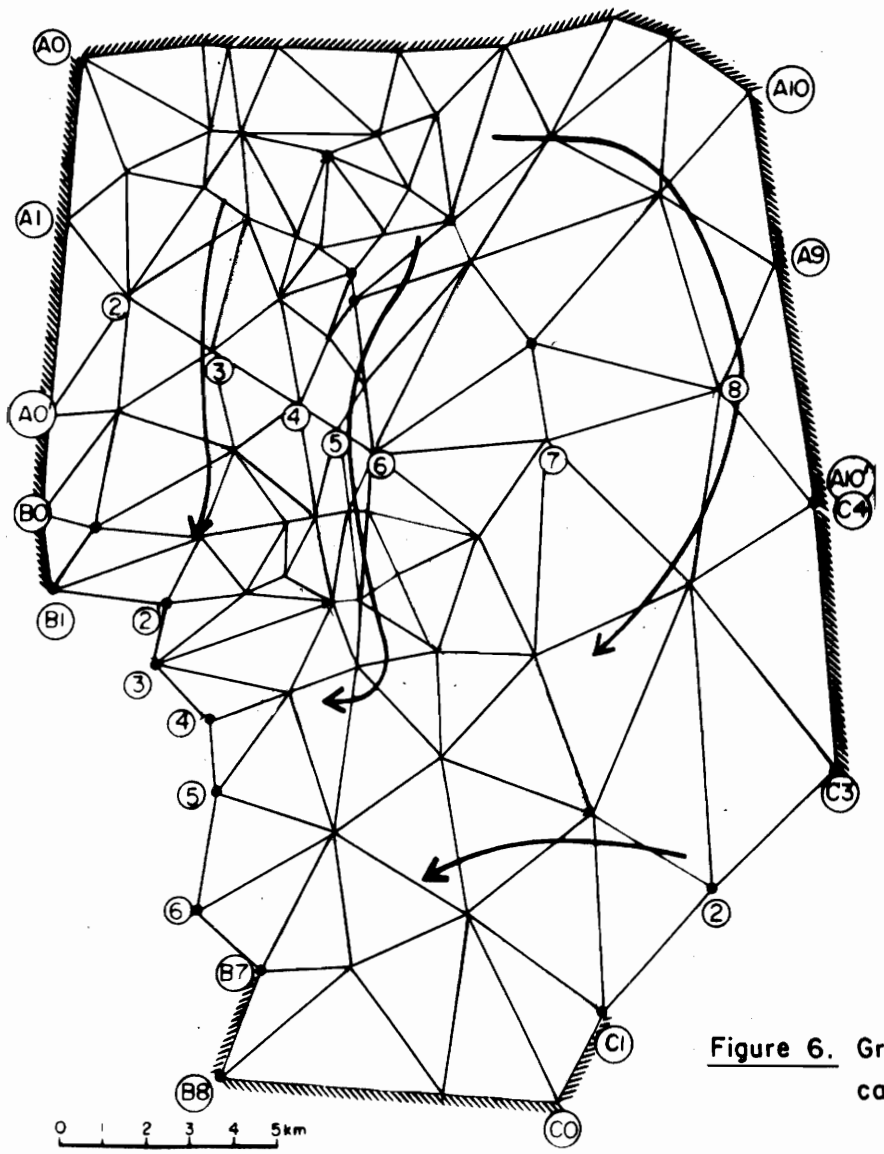

Figure 6. Groundwater area illustrating calculation of flows

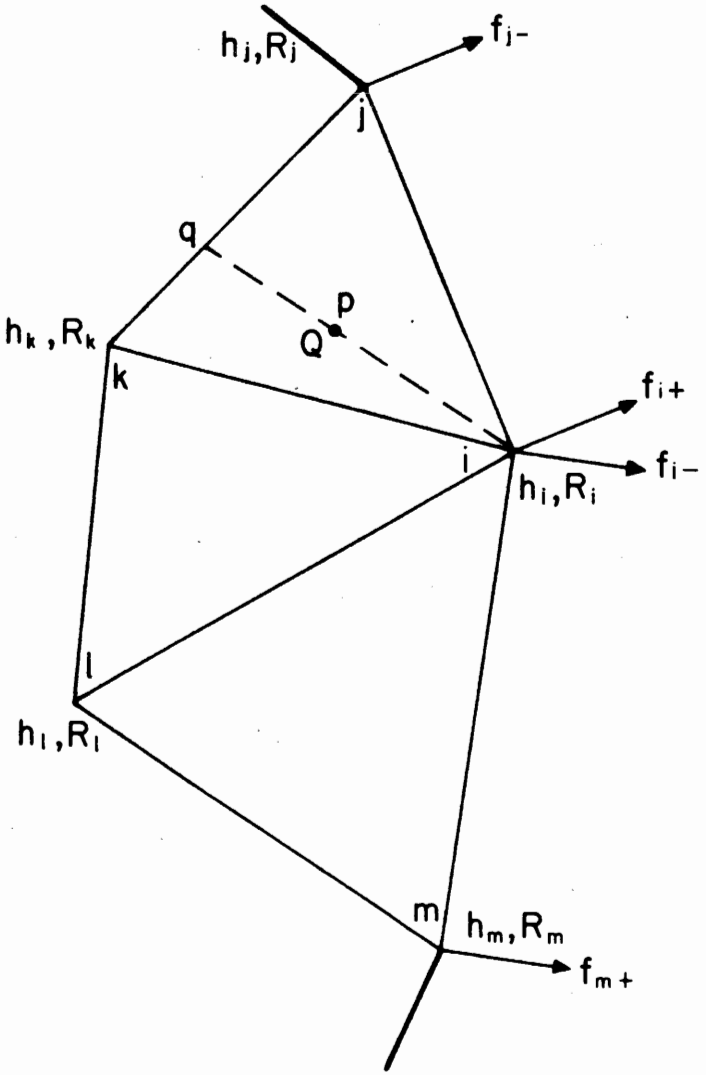

Figure 7. Elements contributing to the Galerkin flow formula

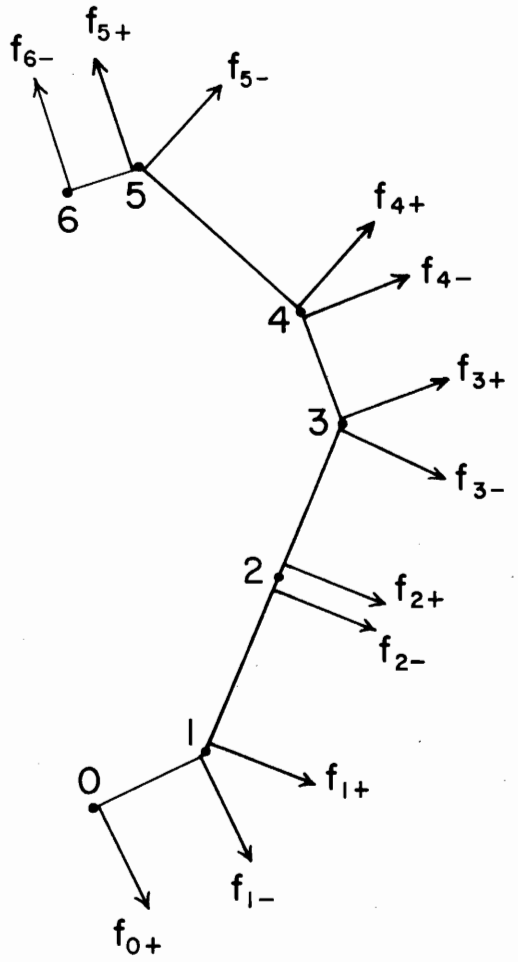

Figure 8. Typical boundary for the Galerkin flow formula

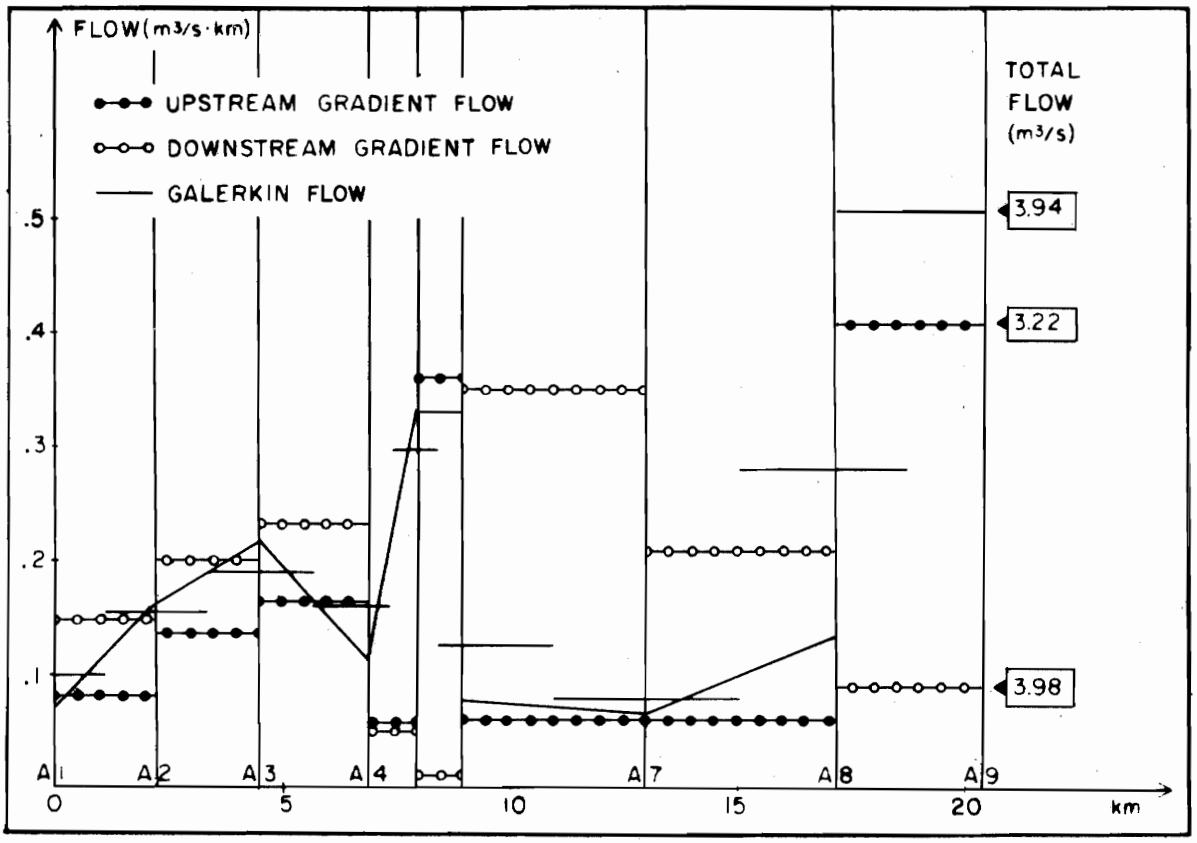

Figure 9. Comparison of flow values across an internal boundary.

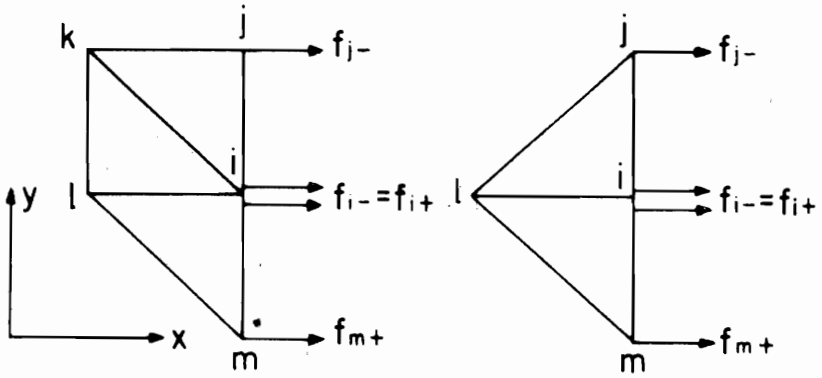

Figure 10. Elements contributing to the Galerkin flow formula in a regular case